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Equilibrium and near-equilibrium turbulent boundary layers

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A well-tested integral method has been used to calculate turbulent boundary-layer development for the distribution of external velocity given by $U \propto x^{-0.255}$. The results suggest that different values of the initial momentum thickness, so long as this is below some critical value, produce a range of equilibrium layers having widely different values of the form parameter G. For values of the initial momentum thickness greater than the critical value, layers are produced which proceed more or less rapidly to separation. These results provide a plausible explanation for conflicting experimental observations made in the past.

Additional calculations for the flows $U \propto x^{-0.15}$ and $U \propto x^{-0.35}$ suggest that, in the first case, a unique equilibrium condition is approached whatever the initial momentum thickness unless this exceeds some critical value; in the second case no equilibrium condition appears possible.

1. Introduction

In the literature on turbulent boundary layers, the subject of equilibrium layers occupies an important place since the self-preserving character of the outer regions of such layers makes it possible to adopt plausible assumptions which greatly simplify the analysis.

In general terms, an equilibrium layer is one in which such non-dimensional parameters as H, c_f, etc., vary only slowly with distance from the origin; more specifically it is one in which the non-dimensional velocity defect $(U-u)/U_\tau$, expressed as a function of y/δ, remains closely invariant with downstream distance.

A measure of the velocity defect is provided by the form parameter G, introduced by Clauser (1954) and defined by

$$G = \int_0^1 \frac{(U-u)^2}{U_\tau^2} d\left(\frac{y}{\delta}\right) \bigg/ \int_0^1 \frac{U-u}{U_\tau} d\left(\frac{y}{\delta}\right).$$

G is simply related to the normal form parameter H by

$$G = \frac{H-1}{H} \bigg/ \left(\frac{c_f}{2}\right)^{\frac{1}{2}}.$$

If G is to remain constant with downstream distance the pressure force acting on the boundary layer must remain in a constant ratio to the skin-friction force, i.e. the parameter $\delta^* \tau_w^{-1} dp/dx$ must remain constant. This parameter has been given the symbol π or β; we shall use π.

Now, there seems to be a considerable degree of unanimity concerning the relationship between π and G. The analysis of Mellor & Gibson (1963, 1966) leads to a definition of this relationship in numerical terms, and an analytic approximation to their result has been given by Felsch (1965). A very similar analytic expression has been suggested by Nash (1965). Numerous comparisons with experiment have shown that Mellor & Gibson's relationship provides a satisfactory correlation between measured values of π and G in equilibrium and near-equilibrium conditions (Felsch 1965; Bradshaw 1966).

Where considerable doubt appears to exist is in the relationship connecting the value of G with the exponent n in the expression $U = cx^n$ describing the distribution of the external velocity. In laminar flow such distributions yield the well-known similar solutions and in turbulent flow the restricted degree of similarity implied by the velocity-defect law. Mellor & Gibson (1966) propose a relationship between G and n which is effectively unique, though there is a minor dependence upon Reynolds number. Townsend's (1961) analysis on the other hand leads to the conclusion that, for the larger negative values of n, two alternative equilibrium layers should correspond to any given value of n. Mellor (1966) professes to have disproved Townsend's result and Bradshaw (1966) argues that the alternative developments arise in Townsend's analysis because of the matching condition imposed between inner and outer solutions for the velocity distribution. He concludes that the balance of probabilities is in favour of a single equilibrium layer corresponding to a given pressure gradient.

On the experimental side, the situation is certainly no clearer. With $n = -0{\cdot}23$, Stratford (1959) set up a layer which was everywhere just on the point of separation, with skin friction effectively equal to zero. In such a layer Mellor & Gibson predict a value of H of 2·35 at a Reynolds number of 10^5, and their predicted profile is in fair agreement with Stratford's measurements. On the other hand, Clauser (1954), with exponents quoted by Townsend (1961) as approximately $-0{\cdot}24$ and $-0{\cdot}25$, obtained layers which were by no means close to separation with values of H of approximately 1·5 and 1·8. At the same time he reported considerable difficulty in establishing equilibrium conditions and referred to the problem of downstream stability, which has not apparently been encountered by subsequent experimenters.

In a very detailed and apparently reliable set of experiments, Bradshaw (1966) obtained equilibrium layers with H values of about 1·6 and 1·4, for values of n of $-0{\cdot}255$ and $-0{\cdot}15$, respectively, and reported no difficulty in setting up these flows. Finally, in experiments in sink flow ($U = -cx^{-1}$), which is admittedly a rather different and special case, Launder & Jones (1969) report not a single equilibrium boundary layer, but a range of layers with an essentially self-preserving character.

These experimental results certainly provide no conclusive evidence in favour of the unique relationship between n and G that follows from the analysis of Mellor & Gibson if the relatively minor effect of the Reynolds number in their analysis is neglected; in fact, if the experimental evidence were taken at its face

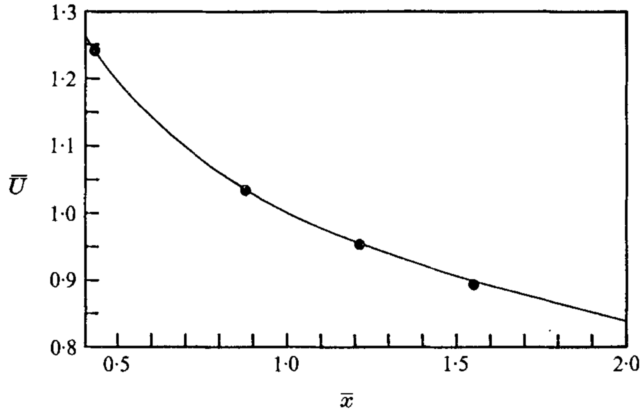


FIGURE 1. Velocity distribution used in most of the calculations. ———, $\bar{U} = \bar{x}^{-0\cdot255}$; ●, Bradshaw's measurements.

value, it would suggest that a whole range of equilibrium layers should exist for values of n close to $-0\cdot25$.

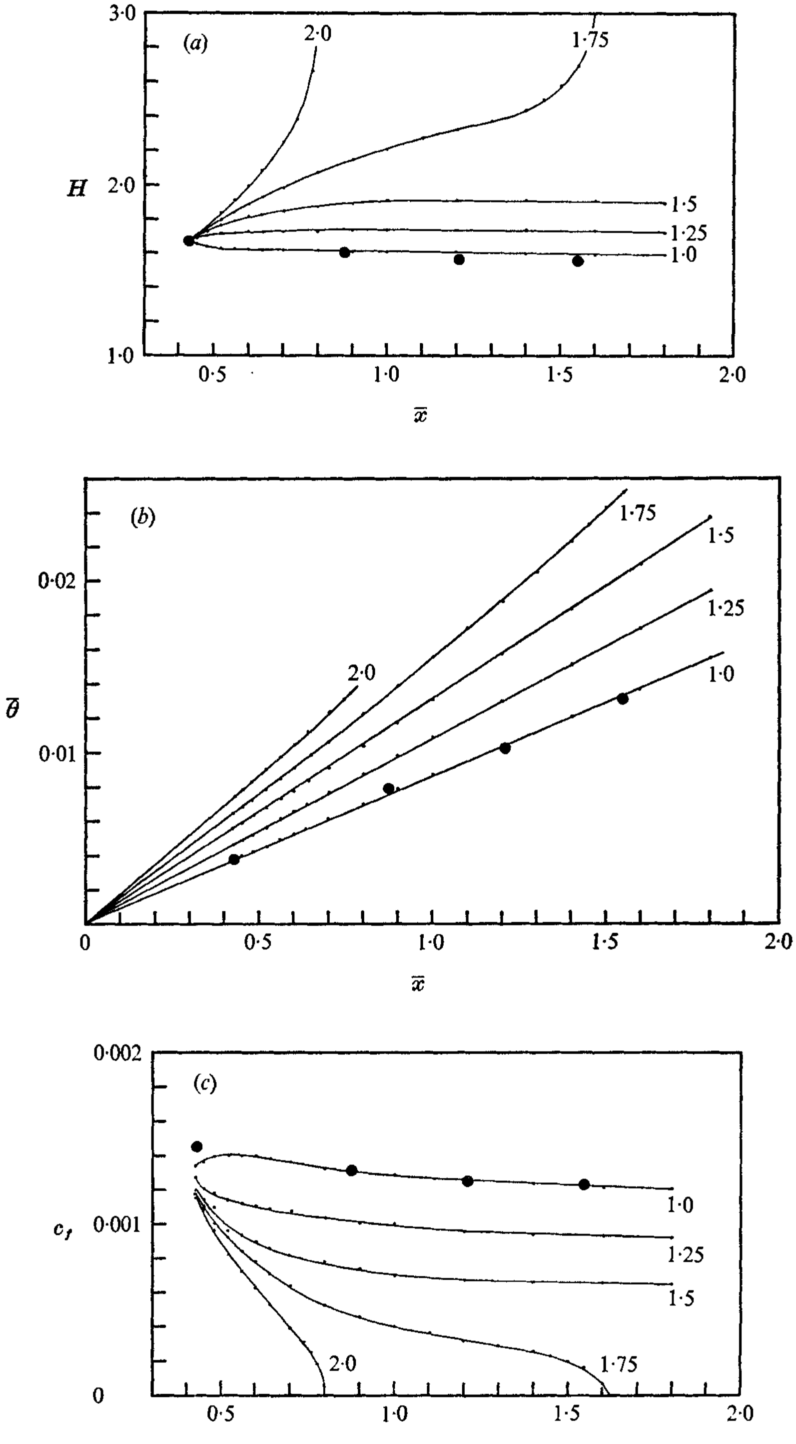
Bradshaw (1966) has pointed out the shortcomings of much of the earlier experimental work and concludes that these are probably sufficient to account for the apparently anomalous results. He supports his conclusion in a later report (Bradshaw 1967) with calculations performed using the method of Bradshaw, Ferriss & Atwell (1967). These suggest that, for $U \propto x^{-0\cdot255}$, a unique equilibrium condition is approached by boundary layers starting with the same value of $U\theta/\nu$ at different streamwise stations. However, it is the present author's opinion that these results cannot be considered conclusive, partly because the calculations are insufficiently precise and comprehensive, and partly because the method of Bradshaw *et al.* fails in some cases to predict the runaway behaviour that is characteristic of the approach to separation.

The results of the present calculations, which have been performed by the integral method of Head & Patel (1970), are at variance with those of Bradshaw and indicate, in fact, that for the particular case $n = -0\cdot255$ a whole range of equilibrium (or pseudo-equilibrium) layers is possible, with values of π and G that satisfy Nash's π–G relation but depend upon the value of $U\theta/\nu$ at some arbitrary initial station. They also indicate that separation ensues if some critical value of $U\theta/\nu$ is exceeded.

Evidence for the trustworthiness of the calculation method used here is provided by the results presented in the original report by Head & Patel (1970), where the equilibrium developments measured by Bradshaw are very accurately predicted, as well as the separating layers measured by Schubauer & Spangenberg.

2. Calculations

Calculations were performed for the more severe of Bradshaw's (1966) two adverse-pressure-gradient cases. As figure 1 indicates, the distribution of the external velocity conforms closely to $\bar{U} = \bar{x}^{-0\cdot255}$, where $\bar{U} = U/U_0$ and $\bar{x} = x/c$.



FIGURES 2(*a*)–(*c*). For legend see facing page.

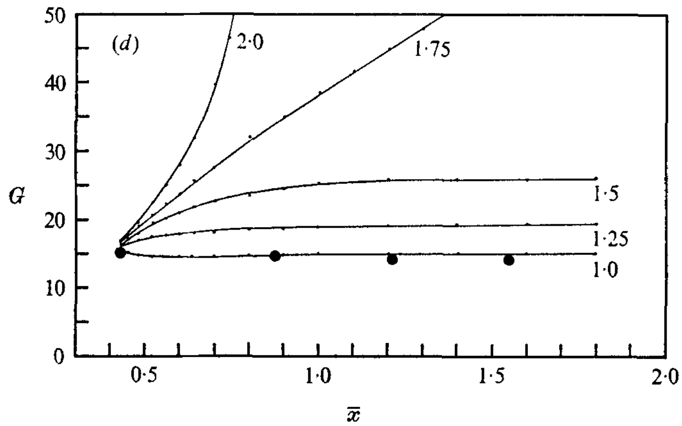


FIGURE 2. Calculated development of (*a*) H, (*b*) $\bar{\theta}$, (*c*) c_f and (*d*) G for different initial values of θ. $n = -0{\cdot}255$. —·—, present calculations; ●, Bradshaw's measurements. Values on curves indicate ratio of initial value of Re_θ to Bradshaw's initial value ($Re_\theta = 14\,600$).

U_0 was given by Bradshaw as 110 ft/s in standard conditions and c was taken as 53·7 in. to eliminate any constant of proportionality.

Using the method of Head & Patel (1970), the development of the boundary layer was calculated for an initial H of 1·667 and initial values of θ which were 1·0, 1·25, 1·5, 1·75 and 2·0 times the initial value measured by Bradshaw. The results of the five sets of calculations, which were performed by hand, are shown in figures 2 (*a*)–(*d*) with Bradshaw's measurements for comparison.

3. Discussion

With the same initial values of H and θ as were measured by Bradshaw, the subsequent developments of H, θ, c_f and G are reasonably well predicted though there are minor discrepancies in H and G.

When we look at the results of the calculations with initial momentum thicknesses 1·25 and 1·5 times Bradshaw's measured value we see no radical change in behaviour, and it is evident that the boundary layer has effectively achieved equilibrium conditions, to within the accuracy of the present method, at some distance downstream. The rate of approach would undoubtedly have been hastened by a more appropriate choice of initial H.

With initial momentum thicknesses 1·75 and 2·0 times the measured value we see a very different type of behaviour, with the boundary layer proceeding to separation, and it seems likely that a highly critical condition should exist for some initial value of the momentum thickness between 1·5 and 1·75 times the experimental value. In this condition the phenomenon of downstream instability reported by Clauser would appear inevitable. Equally, the present calculations would appear to explain the ease with which Bradshaw established equilibrium conditions, since his initial momentum thickness evidently lies well within the range for complete downstream stability.

Bradshaw (1966) has suggested from his interpretation of the turbulent energy equation that such runaway behaviour as that demonstrated by the upper curves in figure 2 (*a*) (and corresponding curves in the other figures) is unlikely to occur, but the present calculations predict such behaviour quite unequivocally, and it is certainly not inconsistent with our general experience of separating flows.

It was at first expected that, with increased initial momentum thickness, the results would represent no more than an approximation to equilibrium conditions, possibly with G substantially constant but with the virtual origin of the boundary layer displaced from the origin of the flow. In fact, however, it will be seen from figure 2 (*b*) that any shift in origin was so small as to be effectively negligible.

So far, then, as the present calculations are concerned the three different layers with initial momentum thicknesses 1·0, 1·25 and 1·5 times the experimental value have equal claim to be considered equilibrium layers, and a range of such layers evidently exists. Whether or not only one of these layers constitutes a true equilibrium layer in some strict academic sense can scarcely be considered relevant to the experimental situation.

4. Subsequent calculations

Following the calculations for $n = -0{\cdot}255$ it was thought worthwhile to obtain additional results for $n = -0{\cdot}15$ and $n = -0{\cdot}35$. These are shown in figures 3 (*a*) and (*b*), and rather contrasting behaviour will be observed in the two cases. For $n = -0{\cdot}15$ it appears that, so long as the initial value of θ is below some critical value, which is in the neighbourhood of 10 times Bradshaw's initial value, there is a slow approach to what is evidently a unique equilibrium condition. On the other hand, for $n = -0{\cdot}35$, there is a slow but sustained increase in the value of G even for the smallest initial value of the momentum thickness considered, which suggests that equilibrium is impossible.

For these computer calculations a fixed initial value of $H^*\,(= (\delta-\delta^*)/\theta)$ was assumed for convenience. This leads to slightly different initial values of H for the different initial momentum thicknesses.

5. Conclusions

The present calculations for $n = -0{\cdot}255$ indicate that a wide range of equilibrium (or pseudo-equilibrium) layers is possible for this particular value of n, with the value of G depending upon the momentum thickness at some arbitrary initial station. Above some critical value of the initial momentum thickness, however, the boundary layer fails to achieve equilibrium and instead proceeds more or less rapidly to separation. These results, while not predicted by accepted theories, may well explain outstanding discrepancies in past experiments.

From the limited results obtained for other values of n it seems likely that unique equilibrium conditions do in fact exist for values of $-n$ less than 0·255 although the approach to equilibrium may be extremely slow if the initial value of the momentum thickness is not suitably chosen. The results for $n = -0{\cdot}35$

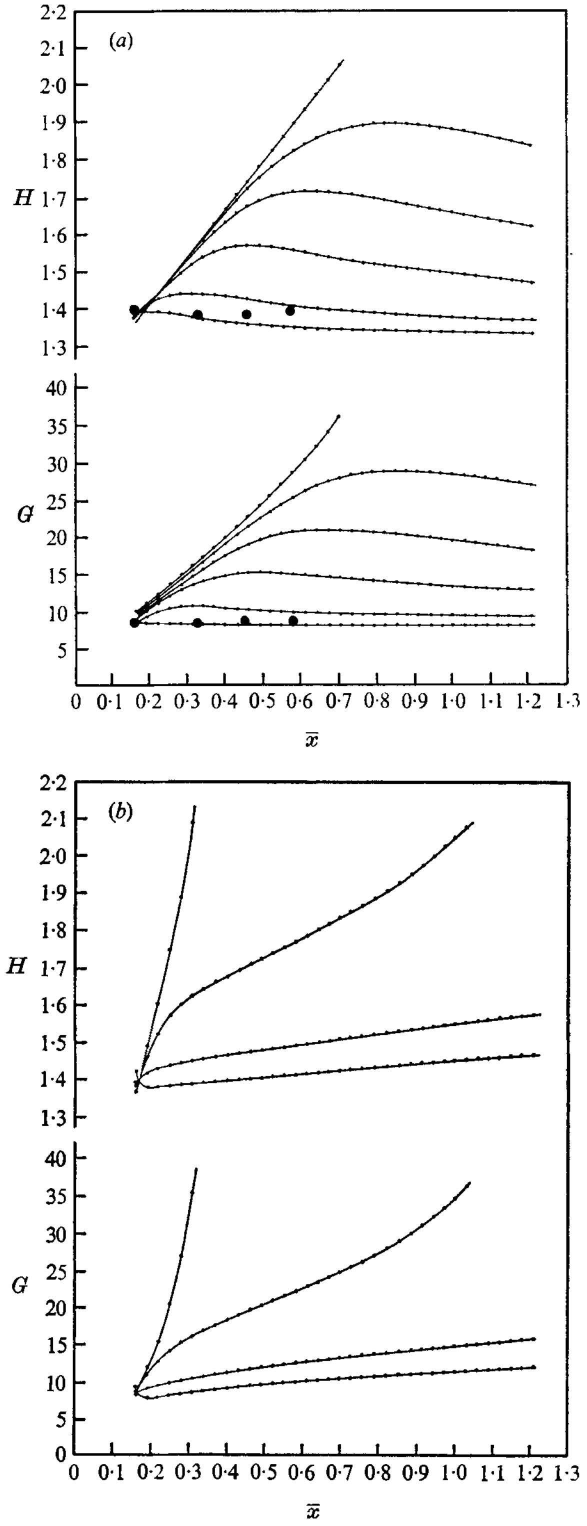


FIGURE 3. Calculated development of G and H for different initial values of θ. (*a*) $n = -0{\cdot}15$. (*b*) $n = -0{\cdot}35$. —·—, present calculations; ●, Bradshaw's measurements. Values on curves indicate ratio of initial value of Re_θ to Bradshaw's initial value ($Re_\theta = 10\,100$).

confirm that equilibrium is impossible for $-n$ greater than 0·255, whatever the initial momentum thickness.

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